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Pricing Exotic Options in a Path Integral Approach

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Abstract. In the framework of Black-Scholes-Merton model of financial derivatives, a path integral approach to option pricing is presented. A general formula to price European path dependent options on multidimensional assets is obtained and implemented by means of various flexible and efficient algorithms. As an example, we detail the cases of Asian, barrier knock out, reverse cliquet and basket call options, evaluating prices and Greeks. The numerical results are compared with those obtained with other procedures used in quantitative finance and found to be in good agreement. In particular, when pricing at-the-money and out-of-the-money options, the path integral approach exhibits competitive performances.

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1. Introduction and motivation

A central problem in quantitative finance is the development of efficient methods for pricing and hedging derivative securities [1, 2, 3]. Although the classical Black, Scholes and Merton model of financial derivatives [4, 5] provides an elegant framework to price financial derivatives, its actual analytical tractability is limited to plain vanilla call and put options and to few other cases. Actually, even if some particular payoffs lead to exact or approximated closed-form pricing formulae [6, 7], these analytical results can be extended to more general payoffs only with difficulty. Hence, there is a need for flexible and fast pricing algorithms, especially when we are interested in pricing options whose payoff at the expiry date depends on the whole path followed by the underlying (i.e. path dependent options). In the past years many approaches have been proposed and the standard numerical procedures adopted in financial engineering involve the use of binomial or trinomial trees, Monte Carlo simulations and finite difference methods [1, 2, 3]. Alternative and more recent algorithms are described, for example, in [8], which has a comprehensive bibliography.

In this paper we extend the path integral approach to option pricing developed for unidimensional assets in [9]. We generalize the original formulation in order to price a variety of commonly traded exotic options. First, we obtain a pricing formula for path dependent options based on multiple correlated underlying assets; second, we improve the related numerical algorithms. Comparisons with standard Monte Carlo simulations, as well as with the results of other numerical techniques known in the literature, are presented. Related attempts to price options and, in particular, exotic options, using the path integral method can be found in [10, 11, 12, 13, 14, 15, 16, 17].

The structure of the article is the following. In Section 2 we trace the derivation of the central pricing formula of our path integral-inspired approach and we describe in Section 3 how to implement it numerically. Details about the computational algorithms used for pricing and numerical results are discussed in Section 4. In this latter, we show that our approach can be efficiently implemented to price a large class of exotic options: Asian, barrier knock out and reverse cliquet. Finally, in Section 5 we compute the Greeks relative to some of the considered options and in Section 6 we draw conclusions and consider possible perspectives.

2. A path integral-based pricing formula

Path integral techniques, widely used in quantum mechanics and quantum field theory, can be useful to describe the dynamics of a Markov stochastic process [18, 19, 20]. In the present paper we are interested in computing mean values of functionals of a D-dimensional Markov stochastic process $S(t)$ corresponding to the price of a set of D underlying assets.

In particular, let us fix a time horizon $T > 0$ and split the time interval $[0, T]$ into $n+1$ subintervals $[T_i, T_{i+1}]$ with $T_0 \equiv 0$, $T_{n+1} \equiv T$ and $T_{i+1} - T_i \doteq \Delta t \doteq T/(n+1)$. Our

aim is to compute the fair price at time T_0 of a European path dependent option with maturity T and whose payoff f is a function of the values $S(0), S(T_1), \ldots, S(T_{n+1})$.

According to the arbitrage-free pricing theory [4, 5, 21], this means that we have to evaluate the mathematical expectation¶

$$\mathbb{E}[f(Z(T_{n+1}), Z(T_n), \ldots, Z_0)] =$$
$$= \int_{\mathbb{R}^{D\times(n+1)}} \mathrm{d}z_{n+1} \cdots \mathrm{d}z_1 \; p(z_{n+1}, z_n, \ldots, z_0) f(z_{n+1}, z_n, \ldots, z_0), \tag{1}$$

where $Z(t) \doteq \log S(t)$ and $p(z_{n+1}, z_n, \ldots, z_0)$ is the joint probability density function (pdf) of the path $\{Z(T_0) = Z_0, \ldots, Z(T_i) = z_i, \ldots, Z(T_{n+1}) = z_{n+1}\}$. In order to compute Equation (1), a straightforward application of standard Monte Carlo estimation theory, would require the sampling of N independent and identically distributed (i.i.d.) paths $\{Z_0, Z^{(l)}(T_1), \ldots, Z^{(l)}(T_{n+1})\}_{l=1,\ldots,N}$. Whenever $S(t)$ (and hence $Z(t)$) is solution of a stochastic differential equation (SDE), this can be done by an Euler discretization scheme of the SDE. Unfortunately, this procedure may be slow, as to compute $Z^{(l)}(T_i)$ we typically need to know $Z^{(l)}(T_{i-1})$, and is not efficient when considering out-of-the-money (OTM) options, because a relevant number of sampled paths may not contribute to the payoff. That is why we look for an alternative formulation of this pricing problem leading to a reduction of the Monte Carlo variance⁺.

2.1. The model

Let us first of all introduce the evolution model we will focus on throughout the rest of the paper. We assume that $S(t)$ satisfies, under the objective probability measure, the following SDE

$$\begin{array}{rcll} dS^k(t)/S^k(t) & = & \mu^k dt + \sigma_k d\bar{W}^k(t) & \forall k = 1, \ldots, D \\ \langle d\bar{W}^i(t), d\bar{W}^j(t)\rangle & = & \rho_{ij} dt & \forall i, j = 1, \ldots, D, \end{array} \tag{2}$$

where the $\mu^k \in \mathbb{R}$ and the $\sigma_k \in \mathbb{R}^+$ represent the mean returns and the volatilities of S^k, respectively, and the $\rho_{ij} \in (-1, 1)$ are the correlations between the components of the Wiener processes $\bar{W}(t)$ ($\rho_{ii} = 1$). This formulation is particularly useful because it only involves financial quantities that can be historically estimated. For instance, ρ_{ij} and σ^k can be evaluated by analyzing the time series of the correlations between different assets' returns, i.e.

$$\begin{array}{rcl} \langle dS^i(t), dS^i(t)\rangle & = & (S^i(t)\sigma_i)^2 dt \\ \langle dS^i(t), dS^j(t)\rangle & = & S^i(t)S^j(t)\sigma_i\sigma_j\rho_{i,j} dt \quad i \neq j. \end{array} \tag{3}$$

However, it is convenient to write Equation (2) in terms of the square root Σ of the variance-covariance matrix $\bar{\Sigma}_{i,j} \doteq \sigma_i\sigma_j\rho_{i,j}$ and of a standard D-dimensional Wiener process W. The square root Σ is defined by relation $\Sigma\Sigma^T = \bar{\Sigma}$ and can be chosen to be a lower triangular matrix. Changing the probability measure from the original

¶ For simplicity, we have included the discount factor $\exp\{-rT\}$ in the definition of the payoff.
⁺ We refer to [2, 22] for a review of standard Monte Carlo variance reduction techniques.

objective measure of Equation (2) to the risk neutral one [21], the stochastic process $Z(t) \doteq (\log S_1(t), \ldots, \log S_D(t))$ satisfies the following equation

$$\begin{cases} dZ(t) = Adt + \Sigma dW(t) \\ Z(0) = Z_0, \end{cases} \tag{4}$$

where the k^{th} entry of A is $A^k = (r - \sigma_k^2/2)$, with r the risk-free interest rate. From Equation (4), we infer that $Z(t)$ is normally distributed with mean $Z_0 + At$ and variance-covariance matrix $\bar{\Sigma}t$. Equivalently, the conditional pdf * $p(z', t'|z, t)$, $t' > t$, is given by

$$p(z', t'|z, t) = \left(\frac{1}{2\pi(t'-t)}\right)^{D/2} \frac{1}{|\det\Sigma|} \exp\left\{-\frac{1}{2(t'-t)}||\Sigma^{-1}(z' - z - A(t'-t))||^2\right\}, \tag{5}$$

where $||\cdot||$ stands for the standard Euclidean norm. Solutions of Equation (4) are Markov processes and, therefore, it is possible to describe their time evolution via a path integral formulation [9].

2.2. The fundamental pricing formula

Thanks to the properties of the chosen model, we are now able to extend the pricing formula given in [9] and propose improved algorithms to evaluate Equation (1). Our approach is essentially based on a sequence of linear changes of the integration variables appearing in Equation (1). This latter expression will then be rewritten in terms of a suitable set of independent random variables.

The definition of conditional probability, together with the Markov nature of the price dynamics, allows us to write the joint probability entering Equation (1) as

$$\begin{aligned} p(z_{n+1}, z_n, \ldots, z_0) \prod_{i=1}^{n+1} \mathrm{d}z_i &= \prod_{i=1}^{n+1} \mathrm{d}z_i \; p_{i-1}(z_i|z_{i-1}) \\ &= \prod_{i=1}^{n+1} \mathrm{d}z_i \left[\left(\frac{1}{2\pi\Delta t}\right)^{D/2} \frac{1}{|\det\Sigma|} \mathrm{e}^{-||\Sigma^{-1}[z_i - (z_{i-1} + A\Delta t)]||^2/2\Delta t}\right], \end{aligned} \tag{6}$$

where we have written the transition densities $p_{i-1}(z_i|z_{i-1}) \doteq p(z_i, T_i|z_{i-1}, T_{i-1})$ explicitly. In order to get rid of the correlation matrix Σ and of the drift A, we perform a first change of variable by setting $z_i = \Sigma(\eta_i + Ai\Delta t)$, $i = 0$ to $n+1$, thus obtaining for the r.h.s. of Equation (6)

$$\text{r.h.s. (6)} = \prod_{i=1}^{n+1} \left[\left(\frac{1}{2\pi\Delta t}\right)^{D/2} \mathrm{d}\eta_i \exp\left\{-\frac{1}{2\Delta t}||\eta_i - \eta_{i-1}||^2\right\}\right]. \tag{7}$$

* By definition, $p(z', t'|z, t)$ is such that the probability for $Z(t')$ taking a value in the D-dimensional hyper-cube dz' centred on z', conditional on $Z(t) = z$, is $p(z', t'|z, t)dz'$.

We work out the quadratic form $\sum_{i=1}^{n+1} ||\eta_i - \eta_{i-1}||^2$ and rearrange terms by introducing the D-dimensional vectors $h_1, \dots, h_n$ such that

$$\eta_i^k = \sum_{j=1}^{n} O_{ij} h_j^k \quad k = 1, \dots D,\ i = 1, \dots, n,$$

where $\boldsymbol{O}$ is the orthogonal matrix that diagonalizes the $n \times n$ tridiagonal matrix

$$\boldsymbol{M} = \begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ 0 & \cdots & -1 & 2 & -1 & 0 \\ 0 & \cdots & \cdots & -1 & 2 & -1 \\ 0 & \cdots & \cdots & \cdots & -1 & 2 \end{pmatrix}. \tag{8}$$

After some tedious, but straightforward, algebra, we obtain

$$\text{r.h.s. (7)} = g(z_{n+1}; z_0) \mathrm{d}z_{n+1} \prod_{i=1}^{n} \mathrm{d}h_i \varrho_i(h_i; z_0, z_{n+1}), \tag{9}$$

where the $\varrho_i(\,\cdot\,; z_0, z_{n+1})$ are D-dimensional Gaussian pdfs with mean $\vartheta_i = [\Sigma^{-1} z_0 O_{1i} + \Sigma^{-1}(z_{n+1} - Ai\Delta t) O_{ni}]/m_i$ and variance $(\Delta t/m_i)\mathbb{1}_{D\times D}$, the $\{m_i\}_{i=1,\dots,n}$ being the eigenvalues of $\boldsymbol{M}$. The function g is defined as

$$g(z_{n+1}; z_0) = \frac{1}{|\det\Sigma|} \left(\frac{1}{\sqrt{2\pi\Delta t \det(\boldsymbol{M})}} \right)^D \\ \exp\left\{ \frac{||\Sigma^{-1} z_0||^2 + ||\Sigma^{-1}(z_{n+1} - A(n+1)\Delta t)||^2 - \sum_{i=1}^{n} ||\vartheta_i||^2/m_i}{2\Delta t} \right\},$$

and from now on we will drop, for simplicity, its dependence on z_0.

Finally, we replace the h_i by setting $h_i = \vartheta_i + \lambda_i \sqrt{\Delta t/m_i}$, thus obtaining the ultimate relationship

$$\text{r.h.s. (9)} = g(z_{n+1}) \mathrm{d}z_{n+1} \prod_{i=1}^{n} \mathrm{d}\lambda_i \rho_G(\lambda_i), \tag{10}$$

where ρ_G is a D-dimensional standardized Gaussian pdf. By means of this sequence of replacements, the expectation (1) can be computed as

$$\mathbb{E}[f(Z(T_{n+1}), \dots, Z_0)] = \\ = \int_{\mathbb{R}^D} \mathrm{d}z_{n+1} g(z_{n+1}) \int_{\mathbb{R}^{D\times n}} \prod_{i=1}^{n} \left(\mathrm{d}\lambda_i \rho_G(\lambda_i)\right) \tilde{f}(z_{n+1}, \lambda_n, \dots, \lambda_1, z_0) \tag{11}$$

where $\tilde{f}(z_{n+1}, \lambda_n, \dots, \lambda_1, z_0) \doteq f(z_{n+1}, z_n, \dots, z_1, z_0)$ with the substitution

$$z_i = \sum_{j=1}^{n} O_{ij} \Sigma \left(\sqrt{\frac{\Delta t}{m_i}} \lambda_j + \vartheta_j \right) + iA\Delta t, \quad i = 1, \dots, n. \tag{12}$$

The reformulation of Equation (1) as in Equation (11) is the core of our pricing technique. In particular, advantages come from having split the $D \times (n+1)$-dimensional

integral into an *external* integration over z_{n+1}, representing the value of the log-price at the maturity, and an *internal* one, which can be thought as the mathematical expectation

$$\mathbb{E}[\tilde{f}(z_{n+1}, \Lambda_n, \dots, \Lambda_1, z_0)] = \int_{\mathbb{R}^{D\times n}} \prod_{i=1}^{n} (\mathrm{d}\lambda_i \rho_G(\lambda_i))\, \tilde{f}(z_{n+1}, \lambda_n, \dots, \lambda_1, z_0),$$

where $\Lambda_1, \dots, \Lambda_n$ are standardized D-dimensional i.i.d. Gaussian variables.

Let us stress that, for each value of z_{n+1}, and since z_0 is known, by means of N i.i.d. Gaussian samples $\left\{(\Lambda_1^{(l)}, \dots, \Lambda_n^{(l)})\right\}_{l=1,\dots,N}$, we can construct the set of log-price paths

$$Z^{(l)}(T_i) = \sum_{j=1}^{n} O_{ij} \Sigma \left(\sqrt{\frac{\Delta t}{m_i}} \Lambda_j^{(l)} + \vartheta_j \right) + iA\Delta t, \qquad (13)$$

having fixed starting and end points $Z^{(l)}(T_{n+1}) = z_{n+1}$ and $Z^{(l)}(T_0) = z_0$. This way of proceeding is typical of path integral techniques, in which functional trajectories with fixed initial and final states are considered [18, 19, 20]. That is why we call our method *path integral pricing*.

3. Computational algorithms

The reformulation of the pricing problem given in Section 2 is useful to price path dependent options: this task has been reduced to the numerical computation of the integrals appearing in Equation (11). In particular, we can adopt the two following procedures

1. We can compute the internal $D \times n$-dimensional integral via Monte Carlo sampling of the Λ_i, and the external D-dimensional one by a deterministic method to be specified. We will call this method *path integral with external integration*. This method turns out to perform well when $D = 1$, as shown in the following.
2. We can perform a pure $D \times (n+1)$-dimensional Monte Carlo integration by properly truncating the integration domain on z_{n+1}. This method will be called *pure Monte Carlo* and is particularly useful when considering OTM options on multidimensional assets.

We provide, in the next two subsections, a more exhaustive insight into the procedures sketched above. We refer to Section 4 for the implemented versions' details and the numerical results.

3.1. Path integral with external integration

This method corresponds to a very precise evaluation of the inner function $\mathbb{E}[\tilde{f}(z_{n+1}, \Lambda_n, \dots, \Lambda_1, z_0)] \doteq \mathcal{E}(z_{n+1})$ for some given values of z_{n+1}. Actually, our aim

is to approximate Equation (11) by a formula like

$$\int_{\mathbb{R}^D} \mathrm{d}z_{n+1}\; g(z_{n+1})\mathcal{E}(z_{n+1}) \approx \sum_{i=1}^{n_{int}} g(z_{n+1}^{(i)})\mathcal{E}(z_{n+1}^{(i)})w_i, \tag{14}$$

with a suitable choice of the integration weights w_i and of the integration points $z_{n+1}^{(i)}$. We can, for example, perform Riemann integration or exploit a quadrature rule [23]. Since $\mathcal{E}(z_{n+1}^{(i)})$ is a non-explicitly solvable mathematical expectation, for each $z_{n+1}^{(i)}$ we estimate it by sampling N i.i.d. from the law of $(\Lambda_1, \ldots, \Lambda_n)$, thus obtaining, at the same time, the associated errors v_i. By virtue of the Central Limit Theorem, the v_i scale with the square root of N, so that the bigger N is, the smaller the error and more precise are the values of $\mathcal{E}(z_{n+1}^{(i)})$. Of course, the choice of the z_{n+1}^{i} influences the final result and has to be done carefully. By means of these coupled outer-deterministic and inner-Monte Carlo integrations, we estimate the price with

$$B_{\pm} \doteq \sum_{i=1}^{n_{int}} w_i g(z_{n+1}^{(i)})\mathcal{E}(z_{n+1}^{(i)}) \pm \sqrt{\sum_{i=1}^{n_{int}} (g(z_{n+1}^{(i)})w_i v_i)^2}, \tag{15}$$

as boundary values for the 68% confidence interval of Equation (11). It is worth noticing that such an error does not include the effect of finiteness of n_{int}. Numerical results providing us with evidence the error due to a finite n_{int} is negligible are reported in Section 4.2.1.

To conclude, let us remark that this procedure, forcing $Z(T_{n+1})$ to take a value in $\{z_{n+1}^{1}, \ldots, z_{n+1}^{n_{int}}\}$, is similar to the variance reduction technique known as *stratified sampling Monte Carlo* [22, 24].

3.2. Pure Monte Carlo

We will show in the next section that when pricing unidimensional assets according to Equation (15), a deterministic choice of final integration points works better than a Monte Carlo one. However, it is known that deterministic integration approaches rapidly lose their competitiveness as the dimension grows. As an alternative, we propose a method based on a pure Monte Carlo integration coupled with the path integral.
First, we choose a function $\Gamma : \mathbb{R}^D \to (0, +\infty)$ such that $\Gamma > 0$ and $\int_{\mathbb{R}^D} \Gamma(z)dz = 1$ and we interpret it as a pdf. Second, we rewrite Equation (11) as

$$\begin{aligned}\mathbb{E}[f(Z(T_{n+1}), \ldots, Z_0)] &= \int_{\mathbb{R}^{D\times(n+1)}} \mathrm{d}z_{n+1}\Gamma(z_{n+1}) \prod_{i=1}^{n} \rho_G(\lambda_i)\mathrm{d}\lambda_i \hat{f}(z_{n+1}, \lambda_n, \ldots, \lambda_1, z_0) \\ &= \mathbb{E}[\hat{f}(Z, \Lambda_n, \ldots, \Lambda_1, z_0)],\end{aligned} \tag{16}$$

where $\hat{f}(z_{n+1}, \lambda_n, \ldots, \lambda_1, z_0) \doteq g(z_{n+1})\tilde{f}(z_{n+1}, \lambda_n, \ldots, \lambda_1, z_0)/\Gamma(z_{n+1})$ and Z is a random variable with Γ as pdf. In other words, we read the pricing formula as the mathematical expectation of a function of $n+1$ independent variables, namely Z and the Λ_i. Our algorithm evaluates Equation (16) by a pure Monte Carlo method extracting N random i.i.d. samples $(Z^{(l)}, \Lambda_1^{(l)}, \ldots, \Lambda_n^{(l)})_{l=1,\ldots,N}$.

This method resembles a standard Monte Carlo simulation of random walks, but there are some subtle differences. First of all, in the random walk case one simulates each path recursively by sampling $n+1$ Gaussian variables without knowing where the considered path will end, while here we want to construct paths leading to a given z_{n+1}. Second, we introduce an asymmetry between z_{n+1} and the λ_i in the sense that z_{n+1} plays a crucial role and we give to it the possibility of being sampled by a non-Gaussian pdf by changing Γ. This turns out to be very useful when pricing OTM options and the Monte Carlo random walk turns out to be not efficient, as shown in the next section.

4. Numerical results and discussion

In this section we report computational issues concerning the pricing of different kinds of path dependent options by means of the path integral procedures discussed in Section 3. We will consider the following types of options: Asian and up-out barrier unidimensional call, unidimensional reverse cliquet and Asian basket call. The dynamics of the underlying assets is assumed to follow Equation (4).

4.1. Algorithms' implemented versions

Before entering into details with numerical results, let us list here which versions of the two general algorithms of Section 3 have been implemented.

(i) *Path Integral with Trapezoidal Integration* (PITP)
This is an algorithm of the type described in Section 3.1. In particular, as deterministic method used to integrate over z_{n+1}, we choose trapezoidal integration with equispaced abscissa [23]. Integration, whenever not differently specified, instead of being performed on $\mathbb{R}^D$, is truncated, for each $k = 1$ to D, to the interval

$$\mathcal{C}_k \doteq [\bar{z}^k - 4\sigma_k\sqrt{T}, \bar{z}^k + 4\sigma_k\sqrt{T}],$$

where $\bar{z}^k \doteq Z_0^k + (r - \sigma_k^2/2)T$ for in-the-money (ITM) and at-the-money (ATM) options and $\bar{z} = \log(K)$ for OTM ones. Tests regarding this choice can be found in Section 4.2.1.

(ii) *Pure Monte Carlo with Flat Sampling* (PIFL)
This is a Pure Monte Carlo algorithm (see Section 3.2) in which we sample the $Z(l)$ of Equation (16) from a uniform pdf on the compact subset $\mathcal{C} \doteq \mathcal{C}_1 \otimes \cdots \otimes \mathcal{C}_D$.

(iii) *Pure Monte Carlo with Truncated Cauchy Sampling* (PICH)
This version of the Pure Monte Carlo method uses, as Γ, a truncated Cauchy pdf centred on $\bar{z}$ and normalized to one on the compact subset $\mathcal{C}$. The particular choice of a Cauchy function is suggested by a simple heuristic reasoning and confirmed by empirical tests. Let f be the payoff of a vanilla (call or put) option with $D = 1$. In order to compute Equation (1), we should integrate a function that is of the form of the product of a $\max(\cdot,\cdot)$ with a Gaussian pdf. As a consequence, there will be

a region $z_{n+1} \in (-\infty, z_{low}] \cup [z_{up}, +\infty)$ in which the integrand is (almost) zero and a region in which we expect it to be slightly wider than a Gaussian one. We will come back to this point at the beginning of Section 4.2.1.

As global benchmarks, we will quote results obtained by means of a standard Monte Carlo random walk (MCRW) technique. In other words, we sample N i.i.d. paths $\{Z_0, Z^{(l)}(T_1), \ldots, Z^{(l)}(T_{n+1})\}_{l=1,\ldots,N}$, built up by iterating, for all $i = 1$ to $n+1$

$$Z^{(l)}(T_i) = Z^{(l)}(T_{i-1}) + A\Delta t + \sqrt{\Delta t}\Sigma\xi_i^{(l)} \quad i = 1, \ldots, N, \tag{17}$$

where the $\xi_i^{(l)}$ are i.i.d. standardized D-dimensional Gaussian random variables.

Furthermore, in order to compare the PITP algorithm with a method similar in spirit, we implemented, in the case $D = 1$, a stratification-like algorithm based on Lévy recursive construction of Brownian motion, the so called Brownian bridge. Details about this testing algorithm, that we will refer to as the Brownian bridge with stratification (BBST), are reported, for completeness, in Appendix A. In some cases, the algorithms are implemented by doubling the number of paths using antithetic variance reduction [2]. Whenever this is done, the corresponding algorithm is pre-fixed by the letters AT.

4.2. Unidimensional asset

4.2.1. Asian option

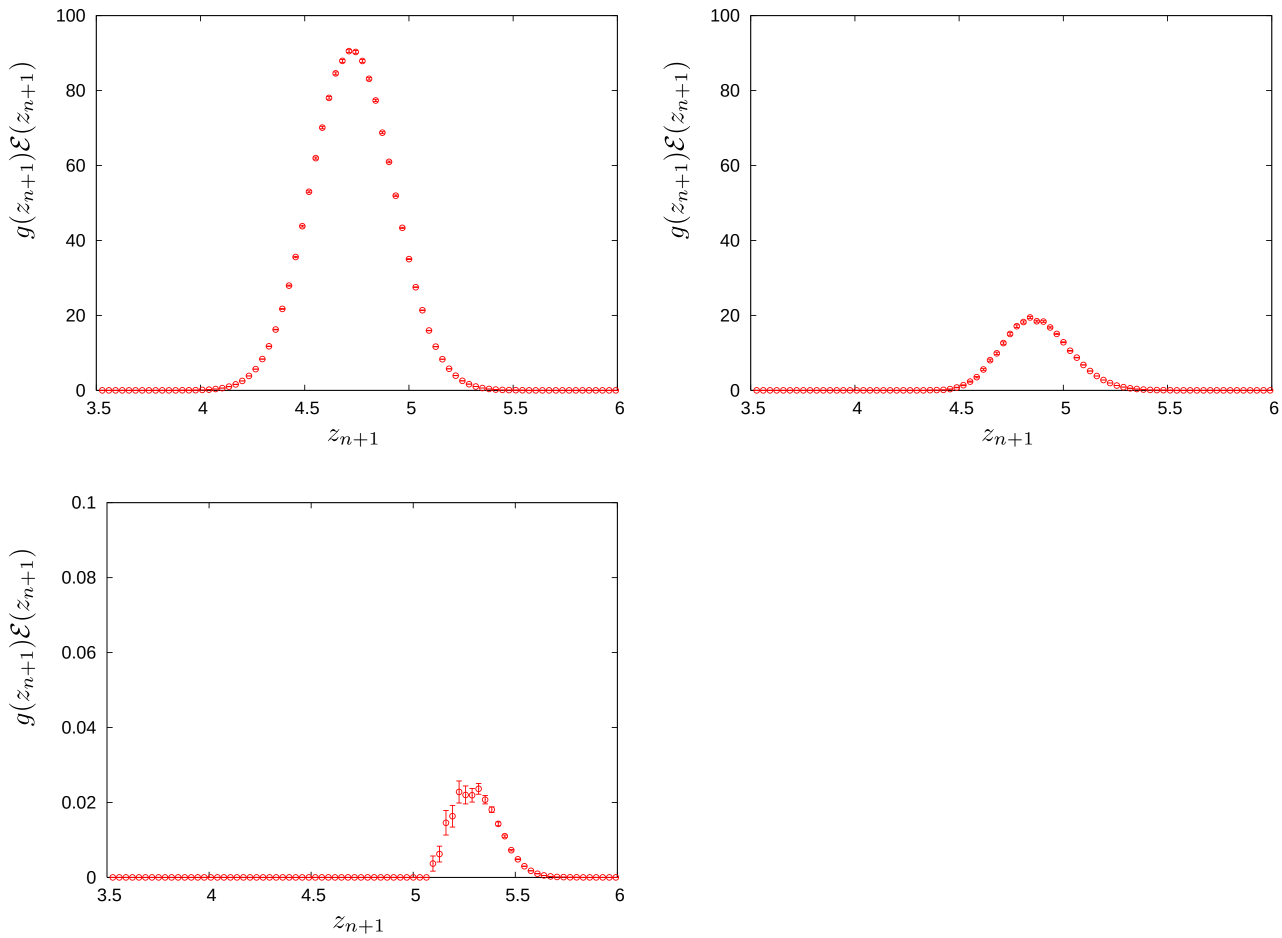
The fair price for a discretely sampled Asian call option on an unidimensional asset is

$$\begin{aligned} \mathcal{O}_A(Z_0) &= \mathbb{E}[f_A(Z(T_{n+1}), \ldots, Z_0)] \\ f_A(Z(T_{n+1}), \ldots, Z_0) &= \mathrm{e}^{-rT} \max\left\{\left(\sum_{i=0}^{n+1} \exp\{Z(T_i)\}\right)/(n+2) - K, 0\right\}, \end{aligned} \tag{18}$$

where K is the strike price and T the maturity. The parameters used in the numerical simulation are: $Z_0 = \log 100$, $r = 0.095$, $\sigma = 0.2$, $t = 0$, $T = 1$ year and $n+1 = 100$. For simplicity, we omit the labels in the definition of Z_0 and σ for all the unidimensional assets. Moreover, we consider $K = 60, 100, 150$ in order to test the performances of our algorithm when the option is ITM, ATM and OTM, respectively.

We justify the choices made about the integration domain discussed in Section 4.1 as follows. Let us approximatively trace the shape of the integrand function $g(z_{n+1})\mathcal{E}(z_{n+1})$ in Equation (11) for Asian call options. In Figure 1 we show the results obtained for an ATM, an ITM and an OTM option. Error bars correspond to one standard deviation Monte Carlo errors. It is particularly interesting to study the support of the integrand function. Actually, for ITM and ATM options, the values of z_{n+1} for which the function is considerably different from zero are more or less centred at $Z_0 + (r - \sigma^2/2)T$, as can be seen from Figure 1. On the other hand, for OTM options, the lower bound is $\gtrapprox \log K$. Hence, we can exploit this property as a rule of thumb to reduce the external integration to a (small) interval significantly contributing to the integral and to eventually perform importance sampling with an appropriate pdf. That is why we adopt particular values for the centre $\bar{z}$ of the integration domain.

Figure 1. Shape of the integrand function $g(z_{n+1})\mathcal{E}(z_{n+1})$ of Equation (11) for an Asian call option, showing how the support and the value of the maximum change when considering in-the-money (top left), at-the-money (top right) and out-of-the-money (bottom left) options.



Let us now come to the discussion of the numerical results. In Table 1 we report option prices and relative numerical errors as obtained by means of our path integral-based algorithms, as well as by the benchmarks. In the PITP and BBST cases, the number of integration points n_{int} is set to 200 and for each point we generate $N = 1000$ random paths. In the cases of MCRW and of pure Monte Carlo path integral with flat (PIFL) or Cauchy (PICH) sampling, the total number of paths is 2×10^5, so that we compare results obtained with the same number of random calls and the comparison is meaningful. In the lower part of the table, we present the results of some of the algorithms improved by the implementation of the antithetic variables technique. We notice that all path integral prices are in very good agreement with the ones obtained via random walks and BBST. As a further cross-check, we compared our path integral predictions with the results of the method developed in [8], finding perfect agreement. ¿From the point of view of variance reduction, the PITP algorithm turns out to be the method that performs best, especially when pricing ATM/OTM options. This

Table 1. Numerical values for an Asian call option price obtained via different algorithms for the parameters $S_0 = 100$, $r = 0.095$, $\sigma = 0.2$, $T = 1$ year and $n+1 = 100$. Errors correspond to one standard deviation.

	ITM[a]		ATM[b]		OTM[c]	
	Price	Error	Price	Error	Price	Error
MCRW	40.830	0.025	6.899	0.019	0.0054	0.0005
BBST	40.824	0.018	6.886	0.015	0.0058	0.0001
PITP	40.811	0.019	6.876	0.015	0.0057	0.0001
PICH	40.767	0.040	6.873	0.019	0.0059	0.0001
PIFL	40.758	0.105	6.880	0.026	0.0057	0.0001
AT-MCRW	40.836	0.002	6.909	0.008	0.0053	0.0003
AT-PITP	40.832	0.004	6.901	0.004	0.0060	0.0001
AT-PICH	40.775	0.031	6.878	0.008	0.0058	0.0001

[a] In-the-money, $K = 60$.
[b] At-the-money, $K = 100$.
[c] Out-of-the-money, $K = 150$.

means that, when the integrand is non-zero only in a region far from $Z_0 + (r - \sigma^2/2)T$, the standard MCRW generates many paths that are not relevant for the mathematical expectation. Furthermore, the PITP and the BBST techniques give essentially the same results, thus confirming our suspicion that the strategy of fixing the end point before generating paths plays the crucial role in the variance reduction. Let us stress that the flat integration gives a worse variance and that PICH and PIFL algorithms perform best only out of the money.

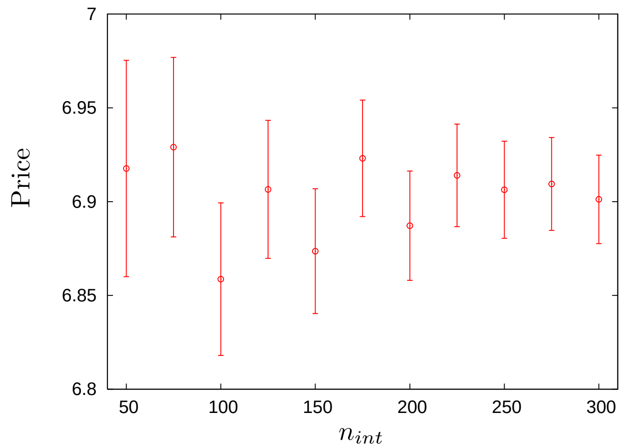
To conclude, let us comment about the estimate of the numerical error for the PITP and the BBST algorithms. Errors in Equation (15) result from the combination of the Monte Carlo errors on each end point, z_{n+1}. To estimate the error associated with the finiteness of n_{int} required by the deterministic integration, we analyzed the stability of the price with respect to the number of integration points. In Figure 2 we show the prices obtained according to this procedure. It can be seen that the fluctuations of the price value due to the choice of n_{int} become negligible with respect to the width of the error bars, as the value of n_{int} increases. This is why we consider the relevant source of numerical error as related to the Monte Carlo part of the integration and we set $n_{int} = 200$ in our simulations.

4.2.2. Up-out barrier options

In this section we consider barrier options of European style, i.e. whose exercise is possible only at the maturity. In particular, we price so called knock-out up options. The payoff at maturity $T_{n+1} = T$ is a functional of the whole path $\{S(s); 0 \leq s \leq T\}$ and has the following expression

$$f_U[S] = \mathrm{e}^{-rT} \max(S(T) - K, 0)\mathbb{1}_{\tau > T} + \mathrm{e}^{-r(\tau - t)} R\mathbb{1}_{\tau \leq T}, \qquad (19)$$

Figure 2. Prices and error bars for an at-the-money Asian call option as a function of the number of points for external integration n_{int}. As n_{int} increases, the error bars decrease ($\sim n_{int}^{-0.5}$), while the fluctuation of prices reduces.



where U is the upper barrier, $\tau = \inf\{s > 0 : S(s) \geq U)$, R is a fixed cash rebate and $\mathbb{I}_{\mathcal{A}}$ is the characteristic function of the set $\mathcal{A}$. In our simulation we set $R = 0$. It is known [24] that, whenever we discretize this continuous time problem, setting

$$f_U(Z_T, \ldots, z_0) = \mathrm{e}^{-rT} \max(\mathrm{e}^{Z(T)} - K, 0) \prod_{i=0}^{n+1} \mathbb{I}_{Z(T_i) < \log U}, \tag{20}$$

we overestimate the price of up-out options. Actually, we do not take into account the possibility that the asset price could have crossed the barrier for some $t \in]T_i, T_{i+1}[$, $i = 0$ to n. A strategy to obtain a better approximation is the following. We check, at each time step $T_i = i\Delta t$, and for all Monte Carlo paths, whether $\exp\{Z^{(l)}(T_i)\}$ has reached the barrier U. If not, we compute the value

$$p_i \doteq \exp\left[-\frac{2}{\sigma^2 \Delta t}(\log U - Z^{(l)}(T_{i-1}))(\log U - Z^{(l)}(T_i))\right], \tag{21}$$

and we extract a random variable from a Bernoulli distribution with parameter p_i. If the result is 1, the barrier value has been reached in the interval $]T_{i-1}, T_i[$ and the price associated to that given path is zero. Otherwise, the simulation is carried on. This technique is widely discussed in [24, 25, 26].

In Table 2 we report prices, with the corresponding numerical errors. We price an ATM option, $K = 100$, and an OTM option, $K = 130$, with $U = 150$ and $U = 200$, all the other parameters as in Section 4.2.1. It is particularly evident that the precision of PITP exceeds that of MCRW, both for ATM and OTM options.

4.2.3. Reverse cliquet options

The last one-dimensional example we consider is represented by the reverse cliquet option, whose payoff function is

$$f_{RC}(Z(T_{n+1}), \ldots, z_0) = \mathrm{e}^{-rT} \max\left[F, C + \sum_{i=0}^{n} \min\left(\frac{S(T_{i+1}) - S(T_i)}{S(T_i)}, 0\right)\right]_{S(T_i) = \exp\{Z(T_i)\}}. \tag{22}$$

Table 2. Numerical values for the price of up-out barrier call options obtained through different algorithms for the parameters $S_0 = 100$, $r = 0.095$, $\sigma = 0.2$, $T = 1$ year and $n + 1 = 100$. Errors correspond to one standard deviation.

	K = 100				K = 130			
	U = 150		U = 200		U = 150		U = 200	
	Price	Error	Price	Error	Price	Error	Price	Error
AT-MCRW	9.087	0.012	12.853	0.015	0.647	0.004	2.353	0.011
AT-PITP	9.088	0.008	12.830	0.001	0.638	0.002	2.336	0.001
AT-PICH	9.099	0.016	12.815	0.014	0.647	0.002	2.333	0.003

Table 3. Reverse cliquet option fair price for $S_0 = 100$, $r = 0.09$, $\sigma = 0.3$ and $T/n = 1/12$ year. Errors are not quoted in [8].

	n = 4		n = 12		n = 24		n = 36	
	Price	Error	Price	Error	Price	Error	Price	Error
AT-MCRW	0.0574	0.0001	0.1223	0.0001	0.1993	0.0002	0.2611	0.0002
AT-PITP	0.0572	0.0001	0.1225	0.0002	0.1992	0.0003	0.2611	0.0003
AT-PICH	0.0572	0.0001	0.1218	0.0002	0.1990	0.0002	0.2606	0.0003
[8]	0.0574		0.1222		0.1990		0.2609	

The option is characterized by the number of periods n, the minimum amount (the *floor*) F, and the maximum payable coupon (the *cap*) C. This option is especially valuable when positive performances are more probable.

We have tested our algorithms by choosing the parameters' values as in [8], whose numerical results for option prices are reported in Table 3, for the sake of comparison. The floor F has been fixed equal to zero, while the cap C is set equal to $n\,c$, with $c = 0.04$. The spot is $S_0 = 100$, $r = 0.09$ and $\sigma = 0.3$. We also change the maturity T by fixing $T/n = 1/12$ year and letting $n = 4, 12, 24, 36$. In Table 3 we show the values corresponding to different values of n. Integration on z_{n+1} for the AT-PITP algorithm is performed on a symmetric interval of width $8\sigma\sqrt{T}$ centred on $Z_0 + (r - \sigma^2/2)T$. Once again we observe accurate path integral pricing and a good agreement between the results of the various algorithms.

4.3. Multidimensional assets

In this section we study the performances of path integral pricing in the case of options on multidimensional assets $S = (S_1, S_2, \ldots, S_D)$. As an example, we price an Asian call option on the basket X whose value at time t is obtained by linearly combining the

values of the components of S:

$$\begin{array}{rcl} X_t & = & \sum_{i=1}^{D} \alpha_i S_t^i \\ \sum_{i=1}^{D} \alpha_i & = & 1 \\ \alpha_i > 0 & & \forall i = 1 \text{ to } D. \end{array} \tag{23}$$

Consequently, we have

$$f_{AD}(Z(T), \ldots, Z_0) = \mathrm{e}^{-rT} \max\left(\frac{1}{n+2} \sum_{j=0}^{n+1} \sum_{i=1}^{D} \alpha_i \exp\{Z^i(T_j)\} - K, 0 \right). \tag{24}$$

In order to compare the multidimensional performance of all the algorithms introduced in Section 3, we need to choose D such that it still makes sense to perform a deterministic integration over $Z(T_{n+1}) = \log S(T)$. However, we expect a gain in competitiveness of pure Monte Carlo integration (PIFL, PICH), as the deterministic approach gradually loses its attractive features as the dimension increases. That is why we choose a three-dimensional correlated asset.

All the tests are performed by setting $r = 0.095$ and considering a maturity of $T = 1$ year with a time discretization of 100 time steps ($n + 1 = 100$). Unless otherwise specified, $\rho_{i,j} = 0.6$ for any $i \neq j$, and $\sigma_k = 0.2$ for all $k = 1$ to $D = 3$. Path integral integration is limited to the compact subset $\mathcal{C}$ described in Section 4.1. In the special case of PITP, we consider 1000 Monte Carlo samples for each end point. It is worth noticing that, if we choose n_1 integration values for each dimension, the total number of required deterministic integration points grows exponentially as n_1^D. Thus, an apparently poor-quality unidimensional integration with $n_1 = 10$ consists indeed in evaluating the integrand function on 10^3 points.

The first test concerns the convergence of the deterministic integration. We set $K = 120$, $S_0 = (100, 90, 105)$ and we compare prices obtained with $n_1 = 6, 8, 10$, i.e. with $216 \cdot 10^3$, $512 \cdot 10^3$, 10^6 total integration points for z_{n+1}. In Table 4 we show the prices thus obtained with their one standard deviation errors, together with the ratio between one standard deviation Monte Carlo error and the price corresponding to $n_1 = 10$ (fourth column), and the percentage difference between Price(n_1) and Price($n_1 = 10$) (fifth column). As in the unidimensional case, changes in prices due to the number of deterministic integration points fall well inside the Monte Carlo 95% confidence interval $[\text{Price} - 1.96\ \text{Error}, \text{Price} + 1.96\ \text{Error}]$.

As a second test, we studied the drawbacks of performing pure Monte Carlo (PIFL, PICH, MCRW) simulations with a small number of Monte Carlo paths (10^4) finding that in these cases MCRW works best, since the path integral fails to efficiently explore the support of the integrand function. Results corresponding to $\rho_{i,j} = 0.8$ for any $i \neq j$, $K = 110$, $\sigma_k = 0.2 + 0.02 \cdot (k - 1)$ and obtained with two different choices for the centre $\bar{z}$ of the integration hyper-cube $\mathcal{C}$ and two different spot S_0 are reported in Table 5. The case $S_0 = (100, 95, 80)$ corresponds to an OTM option, while when $S_0 = (107, 109, 114)$, corresponds to an ATM option. Convergence results are poor as the estimated price depends on the integration region and the Monte Carlo error is large.

Table 4. Prices, one standard deviation Monte Carlo errors, percentage errors and prices' percentage differences of an Asian basket call option, according to the multidimensional PITP algorithm, with $K = 120$. The reference for prices' percentage differences is Price($n_1 = 10$).

n_1	Price	Error	Relative Error	Percentage Price Difference
10	0.306	0.003	0.99%	–
8	0.310	0.005	1.48%	1.23%
6	0.323	0.008	2.38%	5.9%

Table 5. Prices and errors for Asian basket call options according to the algorithms PIFL, PICH, MCRW with a small set (10^4) of Monte Carlo samples. Errors correspond one standard deviation. $K = 110$

	ATM ($S_0 = (107, 109, 114)$)				OTM ($S_0 = (100, 95, 80)$)			
	$e^{\bar{z}}$=(105,110,115)		$e^{\bar{z}}$=(100,120,115)		$e^{\bar{z}}$=(110,110,110)		$e^{\bar{z}}$=(120,120,110)	
	Price	Error	Price	Error	Price	Error	Price	Error
MCRW	7.5	0.1	7.5	0.1	0.83	0.03	0.83	0.03
PIFL	6.3	0.5	7.8	0.6	0.77	0.09	0.78	0.08
PICH	6.8	0.4	8.1	0.4	0.80	0.09	0.56	0.07

Once we take care of choosing a sufficient number of Monte Carlo paths ♯, we compare the performance of our algorithms in the cases of ATM and OTM options. We report the results of such analysis in Table 6, where $S_0 = (100, 90, 105)$ and the strike K is varied to perform ITM/ATM ($K = 100$) and OTM ($K = 140$) pricing. We report results obtained with two differently centred integration intervals, in order to show that the chosen number of samples is enough to guarantee stability of integration to (relatively small) changes of the integration hypercube††. When compared to Table 1, the results shown in Table 6 present some similarities and some differences. As in the unidimensional case, the path integral is still a valuable choice to price OTM options, prices being in agreement with the benchmark MCRW and errors smaller, especially with a Cauchy pdf sampling (PICH). As expected, however, the external deterministic integration (PITP) has effectively lost its attractive properties, PIFL and PICH giving more precise confidence intervals. Let us stress that, when the dimensionality increases, the path integral performance for ATM options worsen. Even if we perform tests with $5 \cdot 10^5$ and 10^6 Monte Carlo samples, whenever we price ITM/ATM options, the path integral method is a bit less precise than the random walk, the central value being nevertheless compatible with the benchmark results. Because the performances

♯ Just take as reference the deterministic integration with the rule of thumb of setting at least 6 integration points for each dimension and 1000 MC paths for each end point, i.e. $6^D \cdot 10^3$ total samples.
††It is important to recall that, if we perform integration on an interval whose spot values are too low, we will have an under-estimation of the price.

Table 6. Prices and errors for Asian basket call options according to the algorithms PITP, PIFL, PICH, MCRW with $n_1 = 6$ and 216000 total Monte Carlo paths. $S_0 = (100, 90, 105)$. Errors correspond to one standard deviation.

	ATM ($K = 100$)				OTM ($K = 140$)			
	$e^{\bar{z}}$=(110,100,110)		$e^{\bar{z}}$=(100,100,100)		$e^{\bar{z}}$=(140,140,140)		$e^{\bar{z}}$=(130,130,130)	
	Price	Error	Price	Error	Price	Error	Price	Error
MCRW	5.29	0.02	5.29	0.02	0.0049	0.0004	0.0049	0.0004
PITP	5.33	0.04	5.28	0.04	0.0051	0.0003	0.0049	0.0003
PIFL	5.37	0.06	5.41	0.07	0.0048	0.0003	0.0048	0.0002
PICH	5.26	0.03	5.28	0.03	0.0048	0.0001	0.0050	0.0001

for OTM options are good, we infer that, to increase the accuracy in pricing high dimensional ITM/ATM options, we should consider an hyper-cube $\mathcal{C}$ wider than four standard deviation.

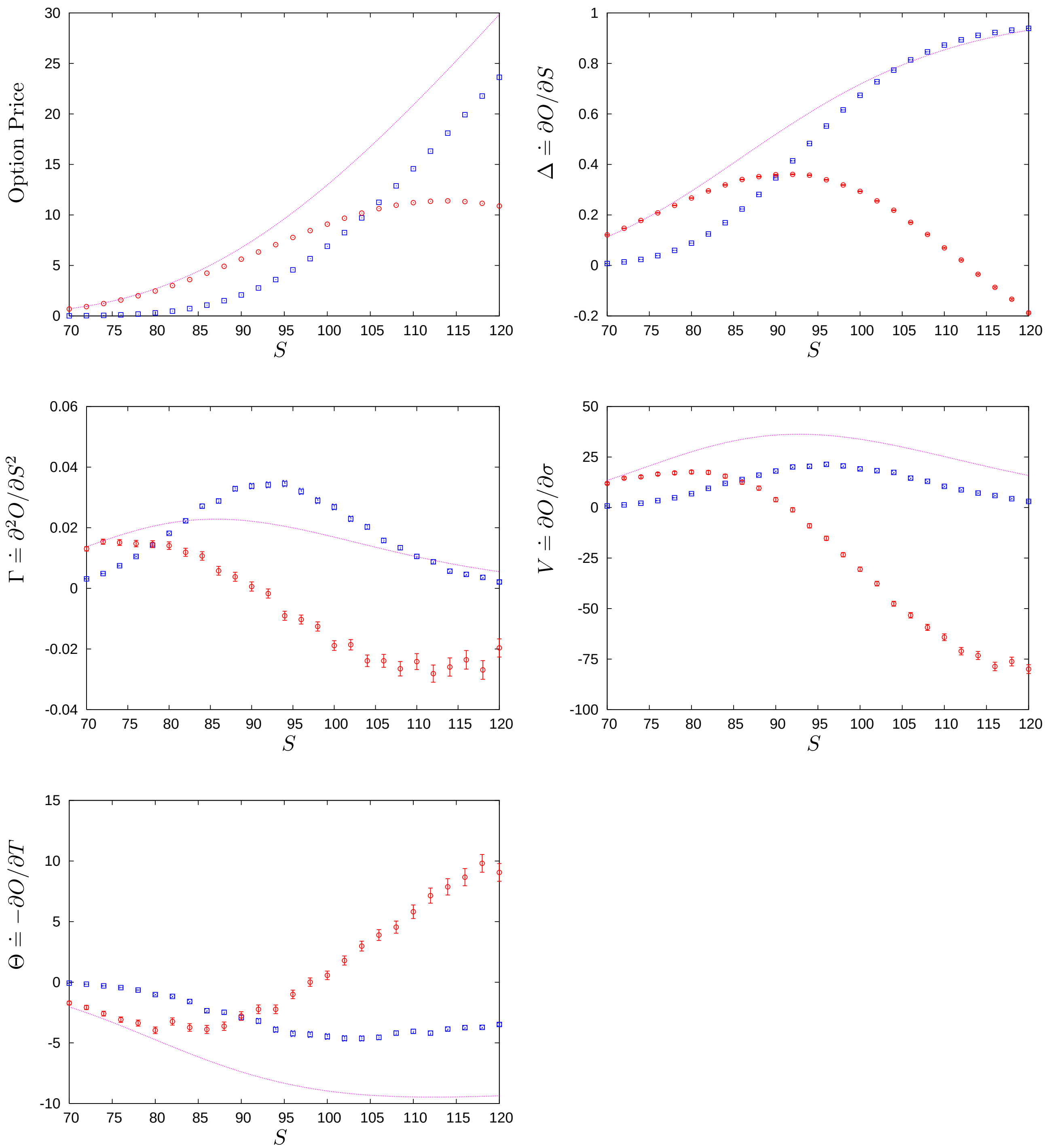
5. Greeks

In the present section, we report our results concerning the computation of the Greeks for unidimensional Asian and barrier knock out options. Actually, it is interesting to compare the numerically estimated exotic Greeks with those implied by the Black and Scholes model for plain vanilla call options. The values of the parameters are $K = 100$, $r = 0.095$, $\sigma = 0.2$, $T = 1$ year and $n+1 = 100$, while for the barrier we choose $U = 150$.

In Figure 3 we show the price of the option and the Greeks delta, gamma, vega and theta as functions of the spot price S. In our approach the Greeks are numerically computed using a finite difference method applied to the option values returned by the path integral with trapezoidal integration. The error bars are obtained by propagating the (one standard deviation) numerical error of the path integral option prices. Therefore, the values of the Greeks are more accurate than those obtained with other approaches when the path integral pricing outperforms existing techniques.

As expected, the qualitative behaviour of prices and Greeks for Asian call options does not differ significantly from that of plain vanilla ones. The shift in prices is due to the fact that in the Asian payoff the role of $S(T)$ is played by the mean value of S along the path. A lower price is therefore a straightforward consequence of the fact that we have $\mathbb{E}[S(T_1) + \cdots S(T_{n+1})]/(n+1) \leq \mathbb{E}[S(T_{n+1})]$. The Greeks do not coincide exactly, but the relevant features, such as the sign of the derivative, are preserved. The situation is completely different for the barrier options. First of all, we expect that for small values of S and with our choice for the max barrier value, $U = 150$, the results overlap the European ones. This can be easily verified from Figure 3 and considered as a check of the consistency of our numerical results. The profile of the price graph is characterized by changes both of the monotonic properties and of the concavity, as shown in Figure 3. This reflects in the change of sign of delta and gamma. The behaviour

Figure 3. Option price and Greeks for plain vanilla (——), Asian (□) and barrier knock out (○) call options.



of vega can be explained by noticing that, for $S \ll U$, an increase of the σ means an increase in the width of the distribution of $S(T)$, that reaches higher values without reaching the barrier, so $\partial O/\partial\sigma > 0$. Conversely, when S and σ grow, $S(T)$ reaches the barrier more frequently and the option loses value. Analogous reasoning applies to theta, i.e. $-\partial O/\partial T$, where the role of σ is played by the maturity T. However, in this

case, the presence of the minus sign in the definition implies $\Theta < 0$ for $S \ll U$ and $\Theta > 0$ otherwise.

6. Conclusions and outlook

In this paper we have shown how the path integral approach to stochastic processes can be successfully applied to the problem of pricing exotic derivative contracts. Numerical results for the fair price and the Greeks of a variety of options have been presented and compared with those obtained by means of other standard (such as the Monte Carlo Random Walk) and non standard (see [8]) approaches employed in quantitative finance. With respect to the original formulation of [9], the method has been generalized in order to cope with options depending on multiple and correlated underlying assets. Concerning options depending on a single asset, it has been shown that the algorithm can provide very precise results, especially when pricing ATM and OTM options. This is due to an appropriate separation of the integrals entering the path integral pricing formula and, more importantly, to a careful simulation of random paths with fixed end points, able to probe the regions contributing to the dominant part of the payoff functions. As far as (multidimensional) basket options are concerned, while the standard Monte Carlo simulation turns out to be more efficient for ITM/ATM options, our approach exhibits better performances for OTM options.

In all the cases, the computational time is essentially the same required by a standard Monte Carlo calculation. The algorithm is general and could be extended to price other types of exotic contracts.

As a future important development, it would be interesting to explore the feasibility of an application of our framework to derivative pricing approaches beyond Black-Scholes and dealing with non-Gaussian features of financial time series, such as Bouchaud-Sornette residual risk minimization [27, 28, 29], stochastic volatility models [30, 31, 32], multi fractal random walks [33, 34], GARCH processes [35] and other generalizations of the log-normal dynamics [36, 37, 38].

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Appendix A. Stratification and Brownian bridge

We describe here the algorithm used to test our hints about the reasons of the good performances of path integral with deterministic integration when pricing path dependent options on unidimensional assets. In order to improve the numerical precision, it is necessary to lead Monte Carlo paths to a region in which the payoff function is different from zero. This is the advantage of performing the external integral in Equation (11) in a clever way and to drive paths toward some fixed end points. The algorithm here described exploits this idea by means of a backward construction of the underlying process (Brownian bridge), instead of using a path integral method. For simplicity, since this method has only been used in the case $D = 1$, we outline here the construction of a unidimensional Brownian bridge only.

The aim is to describe the law of $Z(t)$, with $t \in [0, t']$, once Z_0 and $Z(t')$ are known. By setting

$$Z(t) = Z_0 + \frac{Z(t') - Z_0}{t'} t + \varepsilon(t), \tag{A.1}$$

it is easy to see that $\varepsilon(t)$ is Gaussian with zero mean and variance $t(t' - t)\sigma^2/t'$. It is worth noticing that, while $\mathrm{Law}[Z(t)|Z(t')] \neq \mathrm{Law} Z(t)$, $\varepsilon(t)$ is independent of $Z(t')$. Hence, given Z_0 and $Z(T)$, we construct Monte Carlo paths $Z_0, Z(T_1), \ldots, Z(T)$ by recursively applying Equation (A.1) from $t' = T,\ t = T_n$ down to $t' = T_2,\ t = T_1$. As for the path integral framework of Equation (13) and the standard Euler discretization scheme of Equation (17), sampling a path is equivalent to generating i.i.d. standardized Gaussian variables. The main difference with the path integral method is that here, once we have the l-th Gaussian sample $\{\varepsilon^{(l)}(T_1), \ldots, \varepsilon^{(l)}(T_n)\}$, we are nevertheless forced to simulate the path *recursively*, which means that we cannot infer $Z^{(l)}(T_i)$ if we have not sampled $Z^{(l)}(T_{i+1})$. In the path integral case, instead, extraction is straightforward by means of Equation (13).

Let us now explain how to compute Equation (1) by means of the Brownian bridge. By setting $\bar{f}(z_{n+1}, y_n \ldots, y_1, z_0) \doteq f(z_{n+1}, z_n \ldots, z_0)$ with the replacements

$$z_i = z_0 + \frac{z_{i+1} - z_0}{T_{i+1}} T_i + y_i, \quad i = 1, \ldots, n,$$

we have

$$\begin{aligned}
\mathbb{E}[f(Z(T_{n+1}), \ldots, Z_0)] &= \mathbb{E}[\bar{f}(Z(T_{n+1}), \varepsilon(T_n), \ldots, \varepsilon(T_1), z_0)] \\
&= \int_{\mathbb{R}^D} \mathrm{d}z_{n+1} \rho_{n+1}(z_{n+1}) \int_{\mathbb{R}^{D\times n}} \prod_{i=1}^{n} \left(\mathrm{d}y_i \rho_{\varepsilon,i}(y_i)\right) \bar{f}(z_{n+1}, y_n, \ldots, y_1, z_0) \\
&= \int_{\mathbb{R}^D} \mathrm{d}z_{n+1} \rho_{n+1}(z_{n+1}) \mathbb{E}[\bar{f}(z_{n+1}, \varepsilon(T_n), \ldots, \varepsilon(T_1), z_0)],
\end{aligned} \tag{A.2}$$

where ρ_{n+1} is the pdf of $Z(T)$ and $\rho_{\varepsilon,i}$ is the pdf of $\varepsilon(T_i)$, for $i = 1$ to n.

It is now straightforward to see that Equation (A.2) has the same form as Equation (11). Therefore, we can approximate integration over z_{n+1} as in Equation (14) and evaluate *via* Monte Carlo the inner mathematical expectation.

By means of Equation (14), we are, in some sense, stratifying the domain $\mathcal{D}$ of the random variable $Z(T_{n+1})$ by dividing it into disjoint sub-sets $\mathcal{D}_i$ (here the sub-sets reduce to the points $z^i_{n+1} \in \mathbb{R}$ of Equation (14)). For each sub-set, we then compute the inner integral by forcing $Z(T_{n+1}) \in \mathcal{D}_i$. It is possible to show [24] that this procedure may lead to variance reduction. This way of proceeding has the same qualities and the same limitations as the PITP, yielding less accurate results for multidimensional assets.

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